Relevant vs Irrelevant variables : general discussion

Consider a set of couplings $\{K\}$ that under a RG transformation by a scale factor b become $\{K'\}$. Assuming there exists a fixed point of RG, which we denote by $\{K^c\}$, one may linearize the RG equations close to the fixed point :

$$K'_a - K^c_a = \sum_b T_{ab} (K_b - K^c_b)$$

where the matrix $T_{ab} = \left. \frac{\partial K'_a}{\partial K_b} \right|_{K=K^c}$.

The matrix T is not necessarily symmetric and therefore its left and right eisenvectors may differ, and it may even have complex eigenvalues. Denoting the left eigenvectors of T as $\langle e^i |$, and the corresponding eigenvalue λ^i, consider the variables

$$\langle u^i | = \sum_a \langle e^i_a | (K_a - K^c_a), \text{ so that}$$

$$\langle u'_i | = \sum_{ab} \langle e^i_a | T_{ab} (K'_b - K^c_b)$$

$$= \sum_b \langle e^i_b | (K'_b - K^c_b) \lambda^i$$

$$= \lambda^i \langle u^i |$$

Thus, under RG transformation $\langle u_i |$ transform multiplicatively $\Rightarrow$ λ^i have an exponential dependence on the scale factor b:

$$\boxed{\lambda^i = b^{y_i}}$$

where y^i is the 'RG eigenvalue' of the 'scaling variable' $\langle u^i |$.

For example, in SAW we saw that

$\delta K' = \delta K \times b^{1/\nu}$ (we considered $b=2$ in our example) and thus $y = \frac{1}{\nu}$. The SAW example is too simple since T is a 1×1 matrix. We will encounter more interesting examples soon.

Assuming y_i to be real, there are three possibilities:

(i) $y_i > 0$: repeated RG transformations drives one away from the fixed point. In this case the scaling variable u_i is called relevant. This was indeed the case in the SAW example.

(ii) $y_i < 0$: repeated RG transformations make u_i smaller at every step and it flows to zero at $\{K^c\}$. The corresponding scaling variable is called 'irrelevant'. For example, in the Gaussian random walk, the fourth order cumulant σ_4 flows to zero close to the Gaussian fixed point.

(iii) $y_i = 0$: in this case, the linearized RG equations do not tell us whether the corresponding scaling variable takes us towards or away from the fixed point. There is an interesting possibility of a variable being 'exactly marginal' in which case it doesn't flow at all even when non-linearities are taken into account.

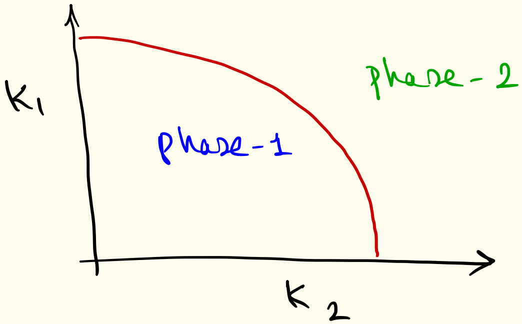
Alternatively, a variable could be 'marginally relevant' or 'marginally irrelevant' depending on whether the higher order terms takes one away from or towards the fixed point.

Broad Picture: Relation to Phase Diagrams

- Fixed points with no relevant scaling parameters correspont to stable phases of matter.
- Fixed points with 'n' relevant scaling parameter correspond to critical points (phase transitions) that require tuning 'n' parameter.

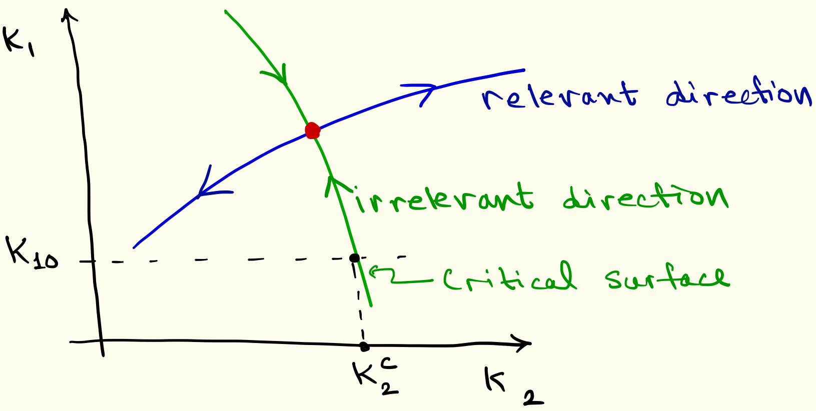
Consider, for example a Hamiltonian with 'n'' total number of parameters, out of which 'n' are relevant and $n' - n$ irrelevant. Therefore, in the phase diagram, there will a $n' - n$ dimensional surface whose points are attraced to the fixed point. The long-distance properties of this 'critical surface' are governed precisely by the fixed point it flows to.

The phase boundary between various phases in a phase diagram is an example of such a critical surface. For example, consider the following phase diagram in a system with two parameters K_1 and K_2:

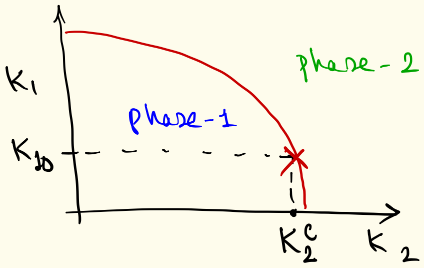


Since reaching the fixed point here requires tuning just one parameter, there is one relevant, and one irrelevant scaling variable corresponding to the fixed point that governs the phase boundary (the red line).

Thus, the RG flow in the vicinity of the fixed point looks like:



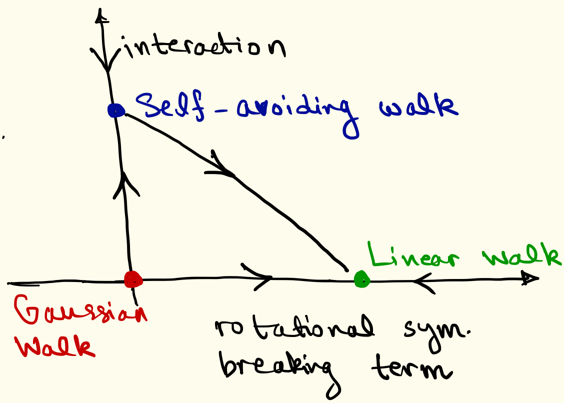
Consider, for example, fixing $K_1 = K_{10}$ and tuning only K_2. In the above RG figure, when K_2 hits the critical surface at $K_2 = K_2^c$, the couplings flow to the fixed point. Therefore, for $K_1 = K_{10}$, the phase boundary occurs at $K_2 = K_{2c}$. This precisely corresponds to the crossed (×) point in the phase diagram on the left. The fact that all points on the red line flow to the fixed point is the origin of universality.



Going back to the random walks, we encountered several RG fixed points:

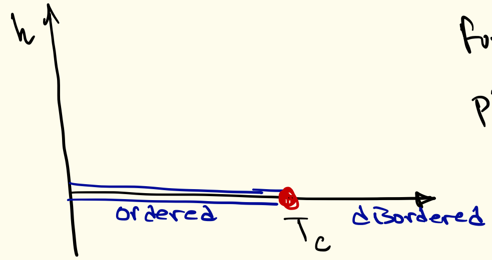
Gaussian random walks, linear walk and self-avoiding walk. Some of these fixed points were stable, and another unstable.

e.g. in $d<4$, in the presence of rotational symmetry, the Gaussian walk is unstable towards the SAW walk:



In the presence of rotational symmetry, therefore one may think of SAW as the stable fixed point, while in the absence of rotational sym, the linear walk is the only stable fixed point.

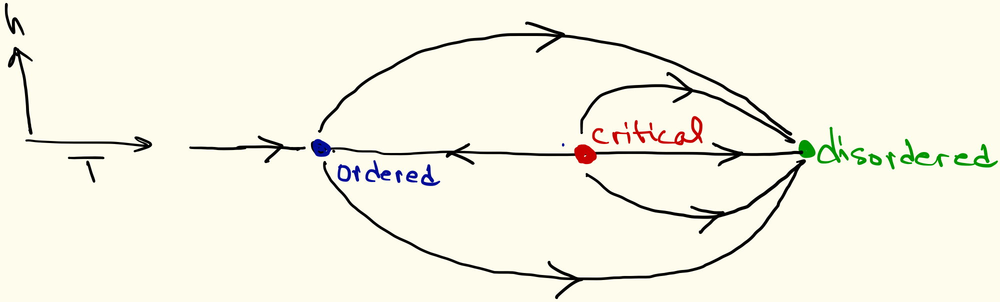
This highlights the importance of symmetries: after imposing certain sym, some relevant variables may not be allowed.



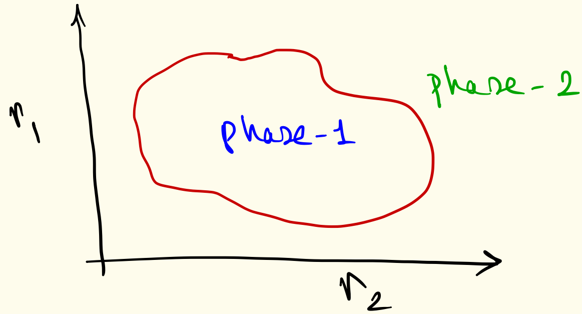
For example, consider the phase digram of the Ising model on the left.

In the presence of Ising sym. ($h=0$), getting to the critical point ($h=0$, $T=T_c$) requires tuning one parameter ($=T$), while obtaining ordered and disordered phases requires no tuning parameter (they are stable phases).

In contrast, in the absence of this symmetry, there is only one stable phase (the disorderd phase) and accessing the critical point requires tuning two parameters. Thus, the RG flow looks like:



Here's another example:



where r_1, r_2 are some tuning parameters.

Again, going from phase-1 to phase-2 requires only one tuning parameter (some combination of r_1, r_2 depending on the direction of approach). Therefore, accessing the phase transition between them requires one tuning parameter. Thus, the RG flow is: